USE OF THE PONTRYAGIN MAXIMUM PRINCIPLE TO SOLVE AN INVERSE HEAT CONDUCTION PROBLEM

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A numerical solution of an inverse problem in heat conduction theory is obtained using a gradient method based on the discrete Pontryagin maximum principle.

In many investigations in experimental thermophysics the need arises to solve the so-called inverse heat conduction problem (IHCP). For example, to determine the heat transfer coefficient

$$\alpha = \frac{\Phi_c(l,\ \tau)}{\theta_c(l,\ \tau) - t_f},$$

where Φ_c is the heat flux; θ_c is the temperature of the inner wall surface; t_f is the mean calorimetric fluid temperature. Direct measurement of Φ_c and θ_c is difficult in the majority of cases, while it is easy to measure the temperature of the outer surface θ_* of the channel wall and the heat flux Φ_*. Therefore, the problem is to determine θ_c and Φ_c by solving the heat conduction equation (for simplicity we shall restrict consideration to an infinite plate of thickness l)

$$\frac{\partial \theta}{\partial \tau} = a \frac{\partial^2 \theta}{\partial x^2}, \quad 0 \leqslant x \leqslant l; \quad 0 \leqslant \tau \leqslant T \tag{1}$$

with the known boundary conditions

$$\theta|_{x=0} = \theta_*(\tau), \tag{2a}$$

$$-\lambda \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = \Phi_*(\tau). \tag{2b}$$

However, the problem is incorrect in this formulation, and it is known that considerable difficulties arise in solving it. Moreover, direct measurement of temperatures and heat fluxes on the outer plate surface require a certain time, during which the plate acquires (or gives up) heat from inside, without appreciable change of the temperature at a point on the outer surface. Since the temperature at such a point is a function of the total heat flux, which varies with time, the initial indeterminacy does not allow a unique solution of the IHCP from the empirically obtained temperature values, using methods of solution which determine the unknown quantities Φ_c, θ_c at each ambient time instant.

Several approaches to solution of an IHCP are known in the literature. Operational methods are used in [1-3]. In [4-6] a solution is found in series form, the coefficients depending on time, and determined in terms of the n-derivatives of the measured temperature $\theta_*(\tau)$ and the heat flux $\Phi_*(\tau)$, i.e., infinite differentiability of the boundary conditions is assumed. However, it is known that if $\theta_*(\tau)$ and $\Phi_*(\tau)$ have been determined experimentally, it is not possible to determine the higher derivatives accurately, a fact which has an appreciable bearing on the practical value of the solution obtained [4-6]. Frank [7] uses a method of least squares to solve an IHCP. A defect of the method is the need to represent the heat flux in the form $\Phi_c = \sum_{n=0}^{\infty} a_n \tau^n$, and also the difficulties involved in solving (by the least-squares method) an over-defined linear system of high order relative to a_n. Reference [8] uses the Tikhonov regularization method to solve an IHCP.

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In order to bring out the difficulties in solving an IHCP, we shall use an iterative method of solution. We shall consider a finite-difference approximation of Eqs. (2b) and (1):

$$3\theta_0^{\tau+\Delta\tau} - 4\theta_1^{\tau+\Delta\tau} + \theta_2^{\tau+\Delta\tau} = 0, \tag{3}$$

$$\theta_{n+1}^{\tau+\Delta\tau} - B\theta_n^{\tau+\Delta\tau} + \theta_{n-1}^{\tau+\Delta\tau} = -F_n^\tau, \quad n = 1,\ 2,\ \ldots,\ N = 1, \tag{4}$$

where $B = 2 + (\Delta x^2/\Delta\tau a)$; $F_n^\tau = (\Delta x^2/\Delta\tau a)\,\theta_n^\tau$. We note that, although the system of equations (3) and (4) is unfactorized, it is easy to generalize the iterative method of solution, as follows.

Straight-line steps:

$$\beta_1 = 2 - \frac{B}{2};\quad z_1 = \frac{F_1^\tau}{2};\ \beta_{n+1} = \frac{1}{B-\beta_n};\quad z_{n+1} = \beta_{n+1}(z_n + F_n^\tau), \tag{5}$$

where β_n, z_n are step coefficients, corresponding to the problem with boundary condition (2b). With boundary condition (2a) and taking into account Eq. (4), we have

$$\beta_1' = 0;\quad z_1' = \theta_*^{\tau+\Delta\tau};\quad \beta_{n+1}' = \frac{1}{B-\beta_n'};\ z_{n+1}' = \beta_{n+1}'(z_n' + F_n^\tau).$$

The relations for reverse steps are:

$$\theta_n^{\tau+\Delta\tau} = \beta_{n+1}'\theta_{n+1}^{\tau+\Delta\tau} + z_{n+1}'; \tag{6a}$$

$$\theta_n^{\tau+\Delta\tau} = \beta_{n+1}\theta_{n+1}^{\tau+\Delta\tau} + z_{n+1}. \tag{6b}$$

Therefore, to determine the temperature with $x = l$, the matter reduces to solution of the system of two equations (6a), (6b) with $n = N - l$, whose determinant is $\Delta = \beta_N' - \beta_N$. It is easy to see that the value of β_n tends to the limiting value $\beta = (B - \sqrt{B^2 - 4})/2$ as n increases, independently of the initial approximation $[\beta_1 = 2 - (B/2)$ or $\beta_1' = 0]$. Since $\beta_N' - \beta_N \to 0$ as $N \to \infty$, the accuracy in finding θ_N from Eqs. (6a) and (6b) decreases with increase of N. The method suggested in the present paper for solution of the IHCP is the Pontryagin discrete maximum principle [9], and an example of solution by this method is given.

In order to use methods from the theory of optimum control to solve the IHCP, we shall formulate the problem in an equivalent variational form. To do this, we consider $\theta_c(l, \tau) = \theta_N(\tau)$ as a control function of a process described by the heat conduction equation with boundary condition Eq. (2) (for simplicity we shall put $\Phi_* = 0$ below), and shall determine $\theta_N(\tau)$ from the condition that the functional

$$I = \int_0^T \frac{M}{2}\,[\theta(0,\tau) - \theta_*(\tau)]^2\,d\tau \tag{7}$$

should be a minimum, or in finite-difference form:

$$I^{\tau+\Delta\tau} = I^\tau + \frac{M\Delta\tau}{2}\,(\theta_0^{\tau+\Delta\tau} - \theta_*^{\tau+\Delta\tau})^2, \tag{8}$$

where M is a weighting coefficient. Here it is assumed that the control function $\theta_N(\tau)$ belongs to some region $\theta_{min} \le \theta_N \le \theta_{max}$. We shall construct the associated system of equations for Eqs. (3), (4), and (8):

$$p_1^{\tau-\Delta\tau} = \frac{p_2^{\tau-\Delta\tau}}{\frac{2}{3} + \frac{\Delta x^2}{\Delta\tau a}} + \frac{\frac{\Delta x^2}{\Delta\tau a}\left[p_1^\tau + \frac{4}{3}\,\Delta\tau\,M(\theta_0^{\tau-\Delta\tau} - \theta_*^{\tau-\Delta\tau})\,p_N\right]}{\frac{2}{3} + \frac{\Delta x^2}{\Delta\tau a}},$$

$$p_3^{\tau-\Delta\tau} - Bp_2^{\tau-\Delta\tau} + \frac{2}{3}\,p_1^{\tau-\Delta\tau} = -\frac{\Delta x^2}{\Delta\tau a}\,p_2^\tau + \frac{\Delta x^2}{3a}\,M(\theta_0^{\tau-\Delta\tau} - \theta_*^{\tau-\Delta\tau})\,p_N,$$

$$p_{n+1}^{\tau-\Delta\tau} - Bp_n^{\tau-\Delta\tau} + p_{n-1}^{\tau-\Delta\tau} = -\frac{\Delta x^2}{\Delta\tau a}\,p_n^\tau,\quad n = 3,\ 4,\ \ldots,\ N-2, \tag{9}$$

$$p_{N-1}^{\tau-\Delta\tau}B = \frac{\Delta x^2}{\Delta\tau a}\,p_{N-1}^\tau + p_{N-2}^{\tau-\Delta\tau}$$

and shall use a gradient method, similar to that in [10]. For this purpose we assign some zero-order approximation for the desired temperature $\theta_N^{(0)}(\tau)$. Using the iterative method of Eqs. (5) and (6b) to solve the system of equations (3), (4), and (8) from $\tau = 0$ to $\tau = T$, we find and store the quantity $M[\theta_0(\tau) - \theta_*(\tau)]$ at several time instants τ, and for $\tau = T$ we determine the functional F and $p_n(T)$ $(n = 1,\ 2,\ \ldots,\ N)$:

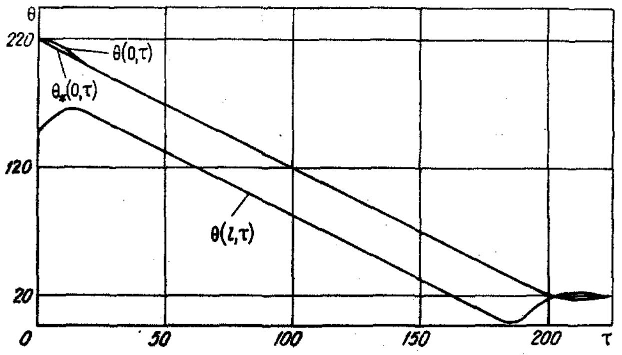

Fig. 1. Solution of an inverse heat conduction problem $\theta(l, \tau)$ for a linear variation of measured temperature $\theta_*(0, \tau)$ and heat flux $\Phi_* = 0$.

$$F = \sum_{n=1}^{N-1} \frac{M_n}{2} [\theta_n(T) - \theta_*(T)]^2 + \frac{M_N}{2} I^2(T), \tag{10}$$

$$p_n(T) = M_n [\theta_n(T) - \theta_*(T)], \quad p_N(T) = M_N I(T), \tag{11}$$

where M_n, M_N are also weighting coefficients.

With the end conditions found at $\tau = T$ for $p_n(T)$, by solving the associated system, Eq. (9), from $\tau = T$, $\tau = 0$, using an iterative method, we find $p_n(\tau)$, and also $\partial H/\partial\theta_N = p_{N-1}(\tau)a/\Delta x^2$, and the maximum of $\partial H/\partial\theta_N \equiv \partial H_0/\partial\theta_N$, in absolute magnitude. Here the iteration coefficients take the values

$$\beta_2 = \frac{1}{\frac{2}{3} + \frac{\Delta x^2}{\Delta\tau a}},$$

$$z_2 = \frac{\frac{\Delta x^2}{\Delta\tau a}\left[p_1^\tau + \frac{4}{3}\Delta\tau M(\theta_0^{\tau-\Delta\tau} - \theta_*^{\tau-\Delta\tau}) p_N\right]}{\frac{2}{3} + \frac{\Delta x^2}{\Delta\tau a}};$$

$$\beta_{n+1} = \frac{1}{B - C_n\beta_n}, \quad z_{n+1} = \beta_{n+1}(C_n z_n + F_n^\tau), \quad n = 2, 3, \ldots, N-2,$$

where C_n and F_n^τ are determined from Eqs. (9), written in the form:

$$p_{n+1}^{\tau-\Delta\tau} - Bp_n^{\tau-\Delta\tau} + C_n p_{n-1}^{\tau-\Delta\tau} = -F_n^\tau, \quad n = 2, 3, \ldots, N-2.$$

The inverse iteration relations are

$$p_{N-1}^{\tau-\Delta\tau} = \frac{z_{N-1} + \frac{\Delta x^2}{\Delta\tau a} p_{N-1}^\tau}{B - \beta_{N-1}};$$

$$p_n^{\tau-\Delta\tau} = \beta_{n+1} p_{n+1}^{\tau-\Delta\tau} + z_{n+1}, \quad n = 2, 3, \ldots, N-2.$$

For the next iteration we assume

$$\theta_N^{(k+1)}(\tau) = \theta_N^{(k)}(\tau) - \mu^{(k)} \frac{\partial H(\tau)/\partial\theta_N}{|\partial H_0/\partial\theta_N|} \tag{12}$$

and repeat the process until we obtain the given accuracy. Here $\mu^{(k)}$ is the length of the gradient step, assigned initially from physical considerations, and is changed during the iteration, depending on variation of the sign of $\partial H_0/\partial\theta_N$. If $\partial H_0/\partial\theta_N$ changes sign in the k-th iteration, we take $\mu^{(k+1)} = \mu^{(k)}/2$ for the next iteration. If $\partial H_0/\partial\theta_N$ retains its sign for several iterations, we double $\mu^{(k)}$ for the next iteration. By way of example we shall examine solution of an IHCP with a linear law for the variation of measured temperature:

$$\theta_*(\tau) = \begin{cases} 220 - \tau, & \text{if} \quad 0 \leqslant \tau \leqslant 200, \\ 20 \quad , & \text{if} \quad 200 \leqslant \tau \leqslant 220 \end{cases}$$

and the following constants: $l = 0.02$ m; $a = 5.342 \cdot 10^{-6}$ m^2/sec; M = 0.1; $M_n = 0.01$, $M_N = 0.1$; $\mu^{(0)} = 7$; $\Delta\tau = 0.1$; $\Delta x = 5 \cdot 10^{-4}$. Figure 1 shows the results of calculating the temperature variation of the inner wall surface by the above method (12 iterations). The calculation was carried out with a program written for the M-20 machine. The initial approximation used was $\theta_N^{(0)}(\tau) = \theta_*(\tau)$. Figure 1 shows that the solution $\theta_N(\tau)$ is quite monotonic. The calculated $\theta(0, \tau)$ is in good agreement with the given value $\theta_*(\tau)$. There is some discrepancy for small values of time (up to 3°C), due to the effect of indeterminacy in formulating the initial conditions $\theta(x, 0)$ (we assumed $\theta(x, 0) = 220$°C; $0 \leq x \leq l$), and the finite rate of propagation of the heat wave due to the finite-difference method of computation. In this example we naturally consider the initial conditions to be free and we determine them from the condition that the functional (10) be minimized using the algorithm described in [10].

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NOTATION

α, a, λ	are the heat transfer coefficient, thermal diffusivity, and thermal conductivity;
$\Phi_c(l, \tau)$, $\Phi_*(0, \tau)$	are the heat fluxes at the inner and outer plate surfaces;
$\theta(x, \tau)$	is the plate temperature;
$\theta_*(\tau)$	is the measured outer temperature;
$\theta_c(\tau)$	is the desired temperature at the inner plate surface;
τ	is the time, sec;
t_f	is the fluid temperature;
x	is the spatial coordinate;
l	is the plate thickness;
$\Delta\tau$, Δ x	is the pitch of mesh in time and space;
p	is the associated variable.

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